Trigonometric series with a given spectrum

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To the memory of Salah Baouendi

Abstract

Let $\Lambda \subset \mathbb{R}^n$ be a a closed and discrete set. The vector space consisting of all trigonometric sums whose frequencies belong to Λ is denoted by $\mathcal{T}_\Lambda$. Given an exponent $p \in [1, \infty]$ we say that Λ is $p-$coherent if there exists a compact set $K \subset \mathbb{R}^n$ and for every $R \geq 1$ a constant $C(R)$ such that for every $P \in \mathcal{T}_\Lambda$ one has $(\int_{|x| \leq R} |P(x)|^p \, dx)^{1/p} \leq C(R)(\int_K |P(x)|^p \, dx)^{1/p}$. Every uniformly discrete set is $2-$coherent but in general if $p \neq 2$ a uniformly discrete set is not $p-$coherent. New examples of $p-$coherent sets will be given in this essay.

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1 Four problems on trigonometric sums

Non periodic trigonometric series appear naturally in the study of Dirichlet series on vertical lines. Indeed if $f(s) = \sum_1^\infty a_n \, n^{-s}$ is a Dirichlet series we have $f(\sigma + it) = \sum_1^\infty a_n n^{-\sigma} \exp(-it \log n) = \sum_1^\infty c_n \exp(i\lambda_n \, t)$ with $\lambda_n = -\log n$. Szolem Mandelbrojt was an expert in Dirichelet series. He was the supervisor of Jean-Pierre Kahane and this could explain the interest of Kahane for some of the problems which will be studied below.

A second motivation for studying non periodic trigonometric series is the investigation of the large time behavior of the solutions of the wave equation on a bounded domain or on a compact Riemannian manifold. Let M be a compact Riemannian manifold and Δ be the corresponding Laplace-Beltrami operator. The wave equation on M is

$$\partial_t^2 u - \Delta_x u = 0. \tag{1}$$

Every solution of (1) can be written as a series

$$u(x,t) = \sum_0^\infty [a_k(x) \exp(i\lambda_k t) + b_k(x) \exp(-i\lambda_k t)] \tag{2}$$

where a_k and b_k are eigenfunctions and $-\lambda_k^2$ are the corresponding eigenvalues of the Laplace operator $\Delta : \mathcal{C}^\infty \mapsto \mathcal{C}^\infty$.

In most cases the series (2) is not a periodic function of the time variable. Even if a solution $u(x,t)$ is continuous on $M \times [0,\infty)$ its large time behavior can be quite unexpected and surprising. If it is the case, then there exist continuous solutions of (1) which are not almost periodic.

More generally for $p \in [1,\infty]$, $p \neq 2$, the growth as $t \to \infty$ of $I_p(x,t) = (\int_t^{t+T} |u(x,s)|^p)\, ds)^{1/p}$ can strongly differ from what happens if $p = 2$. This essay focuses on such problems.

Let us fix some notations. The Fourier transform $\mathcal{F}(f) = \widehat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} \exp(-2\pi i x \cdot \xi)\, dx. \tag{3}$$

Once for all $\Lambda \subset \mathbb{R}^n$ is a closed and discrete set. Then Λ can always be ordered as a sequence λ_j, $j \in \mathbb{N}$, tending to infinity. Such a Λ is uniformly discrete if there exists a $\beta > 0$ such that

$$\forall \lambda \in \Lambda,\ \forall \lambda' \in \Lambda,\ \lambda' \neq \lambda \Rightarrow |\lambda' - \lambda| \geq \beta. \tag{4}$$

One writes $P \in \mathcal{T}_\Lambda$ if P is a trigonometric sum whose frequencies belong to Λ :

$$P(x) = \sum_{\lambda \in \Lambda} c(\lambda) \exp(2\pi i \lambda \cdot x). \tag{5}$$

In the case of the wave equation on a compact manifold M, x is replaced by the time variable t, $\Lambda = \{\pm\lambda_k,\ k \geq 0\}$ is a sequence of real numbers, and the expansion (5) is replaced by

$$u(x_0,t) = \sum_0^\infty [a_k(x_0)\exp(i\lambda_k t) + b_k(x_0)\exp(-i\lambda_k t)]. \tag{6}$$

We now return to the general case. Given a closed and discrete set Λ one looks for the compact sets K such that the following property is satisfied:

Property 1.1 *There exists a constant C such that for every $P \in \mathcal{T}_\Lambda$ one has*

$$(\sum_{\lambda \in \Lambda} |c(\lambda)|^2)^{1/2} \leq C\|P\|_{L^2(K)}. \tag{7}$$

This problem has a long history which can be traced back to the late thirties (see A. Ingham [6] or R. Paley and N. Wiener [20]). If (7) holds for a set K then (7) remains true for every compact set L containing K. Given Λ, do minimal sets K exist ? If $\Lambda = \mathbb{Z}$ then $K = [0,1]$ is minimal. If $\Lambda = \mathbb{Z} \cup \{1/2\}$ there does not exist a minimal set K. An interesting example of minimal sets will be given in Theorem 2.1. Needless to say (7) implies that Λ is uniformly discrete.

In our second problem L^2 norms are replaced by L^∞ norms. This second problem was raised by J-P. Kahane in [8].

Property 1.2 *A uniformly discrete set Λ is a "coherent set of frequencies" if there exist a compact set K and a constant C such that for every $P \in \mathcal{T}_\Lambda$ one has*

$$\|P\|_\infty \leq C \sup_{x \in K} |P(x)|. \tag{8}$$

This implies that Λ is uniformly discrete. Here also one is interested in minimal compact sets K. If the trivial case $\Lambda = \mathbb{Z}$ and $K = [0,1]$ is excepted there are no known examples of such minimal sets (see Theorem 7.1).

In our third problem we do not assume that Λ is uniformly discrete. We want to know if there exist a compact set K and a continuous weight $\omega \geq 1$ such that the following property is satisfied:

Property 1.3 *For every $P \in \mathcal{T}_\Lambda$ and every $y \in \mathbb{R}^n$ one has*

$$|P(y)| \leq \omega(y) \sup_{x \in K} |P(x)|. \tag{9}$$

The simplest non trivial example where (9) holds is given by $\Lambda = \mathbb{Z} \cup \sqrt{2}\,\mathbb{Z}$. We then have $\omega(x) = 1 + |x|$. This observation is proved in [17].

More generally given $p \in [1, \infty]$ one would like to investigate the following property:

Property 1.4 *There exist a compact set K and for every $R \geq 1$ a constant $C(R)$ such that for every $P \in \mathcal{T}_\Lambda$ one has*

$$(\int_{|x| \leq R} |P(x)|^p \, dx)^{1/p} \leq C(R)(\int_K |P(x)|^p \, dx)^{1/p} \,. \tag{10}$$

This covers the third property.

Property 1.4 was addressed in [18] on a toy problem. We restricted our attention to the one dimensional case and assumed that $\Lambda = \{k + r_k,\, k \in \mathbb{Z}\}$ where $r_k \to 0$ as $|k| \to \infty$.

Needless to say a trivial solution to these problems is given by a pair (Λ, K) where Λ is a lattice and K is a fundamental domain for the dual lattice Λ^*.

The wave equation on the sphere S^2 which is discussed in [12] provides us with a natural example where property (8) is not satisfied but where (9) holds. For every continuous solution $u(x,t)$ of the wave equation on S^2 one has for every $t \geq 2\pi$ and every $x \in M$:

$$|u(x,t)| \leq C\sqrt{t} \sup_{s \in [0, 2\pi]} |u(x,s)| \tag{11}$$

and this estimate is optimal. This follows from [12] and [18] and explains why there exist continuous solutions of the wave equation on the sphere which are not almost periodic.

2 L^2 estimates

The L^2 theory of non periodic trigonometric sums (in the sense given by Property 1.1) was born in the the thirties. In the sixties this theory has been revitalized by some important applications to control theory [1], [2],[10] and to signal processing [9]. Indeed (7) is equivalent to the following condition, named stable interpolation in [9], [19]:

Property 2.1 *For every square summable sequence $c \in l^2(\Lambda)$ there exists a square integrable function f supported by K whose Fourier transform satisfies $\widehat{f}(\lambda) = c(\lambda)$, $\lambda \in \Lambda$.*

A main breakthrough was achieved by H.J. Landau [9], (1967). While he was working at the Bell Labs in Murray Hill, Landau proved in that Property 1.1 implies that the Lebesgue measure $|K|$ of K satisfies $|K| \geq \overline{\text{dens}}\,\Lambda$. The upper density of Λ will be defined below. Can the converse implication be true ? Does $|K| \geq \overline{\text{dens}}\,\Lambda$ imply Property 1.1 ? It is not even true if one assumes $|K| > \overline{\text{dens}}\,\Lambda$. The simplest counterexample is given by $\Lambda = \mathbb{Z}$ and $K = [0, 1/3] \cup [1, 1 + 1/3] \cup [2, 2 + 1/3] \cup [3, 3 + 1/3]$. The measure of K is 4/3 which exceeds $\text{dens}\,\Lambda$ but (7) is not true since $P \in \mathcal{T}_\Lambda$ is one-periodic and each of the four intervals of K gives the same information on P. The fundamental question raised by Landau's theorem has not been solved until very recently. A first solution was given in [19] and then a second one in [11]. Property 1.1 is well understood if $n = 1$ and if K is an interval: $|K| > \overline{\text{dens}}\,\Lambda$ suffices. But Property 1.1 is mostly open when $n = 1$ and K is a finite union of intervals, or when $n \geq 2$. Then the arithmetical structure of Λ plays a seminal role as it will be illustrated now. For example if $n = 2$, if Λ is a lattice, and if K is a disk, Landau's bound $|K| = \text{dens}\,\Lambda$ cannot be approached. Indeed we have: (7) $\Rightarrow |K| \geq \frac{2\pi}{3\sqrt{3}}\,\text{dens}\,\Lambda$ and $\frac{2\pi}{3\sqrt{3}} > 1$. This gap comes from the fact that the plane cannot be paved with translated copies of a disk. But if Λ is a quasi-crystal Landau's bound can be reached when K is a disk [11]. Another construction can be found in [19]. If Λ is a random perturbation of a lattice this is no longer true [11]. Property 1.1 holds if K is a ball and if its radius R is large enough. To conclude in dimension $n \geq 2$ the minimal ball cannot be deduced from Landau's theorem but depends on a deeper analysis of the structure of Λ.

The role of quasi-crystals in the L^2 theory is illustrated by a remarkable discovery [5]. Let us denote by $\lfloor x \rfloor$ the integral part of a real number x. Then $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x. Let us assume $\alpha > 0, \beta > 0$, $\alpha \notin \mathbb{Q}$, $\alpha + \beta^{-1} \notin \mathbb{Q}$. Let $\lambda_k = k + \beta\{\alpha k\}$, $k \in \mathbb{Z}$, and $\Lambda_\alpha = \{\lambda_k,\, k \in \mathbb{Z}\}$. Then Sigrid Grepstad and Nir Lev proved the following theorem in [5].

Theorem 2.1 *Let K be a finite union of disjoint intervals with end points in $\alpha\mathbb{Z}+\mathbb{Z}$. Then the exponential functions* $\exp(2\pi i\lambda x)$, $\lambda\in\Lambda_\alpha$, *are a Riesz basis of $L^2(K)$ if and only if $|K|=1$.*

Let us observe that $|K|=1$ is Landau's bound. Theorem 2.1 implies that such a K is minimal for Λ_α. Indeed if a compact set L satisfies $L\subset K$, $L\neq K$ then $|L|<|K|=1$ and (7) cannot hold for L by Landau's theorem. Is there an L^p analogue of Theorem 2.1 when $p\neq 2$? We do not know since the proof of Theorem 2.1 given in [5] is using Plancherel formula.

Corollary 2.1 *Let K be a finite union of disjoint intervals with end points in $\alpha\mathbb{Z}+\mathbb{Z}$ with $|K|=1$. Then every $f\in L^2(K)$ can be written on K as*

$$f(x)=\sum_{\lambda\in\Lambda_\alpha} c(\lambda)\exp(2\pi i\lambda x)$$

and we have

$$C_1\|f\|_{L^2(K)}\le (\sum_{\lambda\in\Lambda_\alpha}|c(\lambda)|^2)^{1/2}\le C_2\|f\|_{L^2(K)}. \tag{12}$$

Another example is given by the following construction. Let $\alpha>0$, $\beta>0$ $\alpha\notin\mathbb{Q}$, $\beta|\sin(\pi\alpha)|\in(0,1/2)$, and $\lambda_k^{(\alpha,\beta)}=k+\beta\sin(2\pi\alpha k)$, $k\in\mathbb{Z}$. Let $\Lambda_{\alpha,\beta}=\{\lambda_k^{(\alpha,\beta)},\, k\in\mathbb{Z}\}$.

Theorem 2.2 *The functions* $\exp(2\pi i\lambda x)$, $\lambda\in\Lambda_{\alpha,\beta}$, *are a Riesz basis of* $L^2([0,1])$.

The proof of Theorem 2.2 mimics what was achieved in [5]. The condition $0<\beta|\sin(\pi\alpha)|<1/2$ implies that $\Lambda_{\alpha,\beta}$ is uniformly discrete. Moreover there exists an integer N such that uniformly in k we have

$$\beta\Big|\frac{1}{N}\sum_{k}^{k+N-1}\sin(2\pi\alpha\, j)\Big|\le\theta<1/5. \tag{13}$$

Then S.A. Avdonin's theorem [1] or [2], yields the result. Does an analogue of Grepstad-Lev's theorem hold? Finally if $\beta|\sin(\pi\alpha)|\ge 1/2$, $\Lambda_{\alpha,\beta}$ is not uniformly discrete and $\exp(2\pi i\lambda x)$, $\lambda\in\Lambda_{\alpha,\beta}$, cannot be a basis.

3 Mean periodic functions

As it was proved by Kahane in [8] Property 1.2 and Property 1.3 are extremely appealing problems if the terminology of mean periodic functions is used. Let $\mathcal{C}(\mathbb{R}^n)$ denote the vector space of all continuous functions on $\mathbb{R}^n$, equipped with the topology of *uniform convergence on compact sets*. Given a discrete and closed set Λ it will be assumed that $\mathcal{T}_\Lambda$ is not dense in $\mathcal{C}(\mathbb{R}^n)$ for the topology of uniform convergence on compact sets. This is certainly the case if Λ is uniformly discrete.

Lemma 3.1 *The following two properties are equivalent for a closed and discrete set* Λ :

(*a*) $\mathcal{T}_\Lambda$ *is not dense in* $\mathcal{C}(\mathbb{R}^n)$ *for the topology of uniform convergence on compact sets.*

(*b*) *There exists a compactly supported Radon measure* $\mu \neq 0$ *whose Fourier transform vanishes on* Λ.

This is provided by Hahn-Banach theorem. Beurling and Malliavin proved that (*a*) or (*b*) are equivalent to a remarkable density condition [4] on Λ. There exists a closed and discrete set Λ satisfying conditions (*a*) and (*b*) with an infinite upper density. This will be proved in Section 5. In the case of Dirichlet series $\Lambda = \{\log m,\, m \in \mathbb{N}\}$ has an infinite lower density. Therefore $\mathcal{T}_\Lambda$ is dense in $\mathcal{C}(\mathbb{R}^n)$ for the topology of uniform convergence on compact sets.

Definition 3.1 *A mean periodic function is a function* $f \in \mathcal{C}(\mathbb{R}^n)$ *for which there exists a compactly supported Radon measure* $\mu \neq 0$ *such that* $f * \mu = 0$.

Definition 3.2 *Let us assume that* $\mathcal{T}_\Lambda$ *is not dense in* $\mathcal{C}(\mathbb{R}^n)$ *for the topology of uniform convergence on compact sets. Then the closure of* $\mathcal{T}_\Lambda$ *in* $\mathcal{C}(\mathbb{R}^n)$ *will be denoted by* $\mathcal{C}_\Lambda$.

These definitions and Lemma 3.1 imply that every function $f \in \mathcal{C}_\Lambda$ is a mean periodic function.

Definition 3.3 *The spectrum of a mean periodic function* f *is the set* $S \subset \mathbb{C}$ *of all complex numbers* λ *such that* $\exp(2\pi i x \cdot \lambda)$ *is a limit, for the topology of uniform convergence on compact sets, of linear combinations of translates of* f.

If μ is a compactly supported Radon measure such that $\mu * f = 0$ the spectrum of f is contained in the set of zeros of the Fourier-Laplace transform of μ. If $f \in \mathcal{C}_\Lambda$ its spectrum is contained in Λ.

Here is a more constructive definition of a mean periodic function.

Lemma 3.2 *Let us assume that* (*a*) $f \in \mathcal{C}(\mathbb{R}^n)$ *has a polynomial growth at infinity and* (*b*) $\widehat{f} = \sum_{\lambda\in\Lambda} c(\lambda)\delta_\lambda$ *where* δ_a *is the Dirac measure at* a. *Then* $f \in \mathcal{C}_\Lambda$.

Is the converse implication true? What are the sets Λ enjoying the property that every $f \in \mathcal{C}_\Lambda$ has a polynomial growth at infinity? This is Problem 1.3 and our essay partially answers this natural question. Problem 1.2 has a simple formulation as the following theorem shows.

Theorem 3.1 *Let* $\Lambda \subset \mathbb{R}^n$ *be a closed and discrete set. Then the following two conditions are equivalent:*

(*i*) *Every* $f \in \mathcal{C}_\Lambda$ *is an almost periodic function in the sense of H. Bohr.*

(*ii*) *Property 1.2 is satisfied.*

This is proved in [16].

4 Wild sets of frequencies

Definition 4.1 *A closed and discrete set $\Lambda \subset \mathbb{R}^n$ is 'wild' if* (9) *does not hold whatever be the compact set K and the weight ω.*

In the one dimensional case a simple criterion is given by the following theorem:

Theorem 4.1 *Let Λ be a closed and discrete set of real numbers. Then Λ is wild if and only if one of the two following conditions is satisfied*

(*a*) *There exists a sequence $P_j \in \mathcal{T}_\Lambda$ such that $P_j(0) = 1$ and such that for every compact set K contained in $(0, \infty)$ we have*

$$\sup_{y \in K} |P_j(y)| \to 0,\ j \to \infty. \tag{14}$$

(*b*) *There exists a sequence $Q_j \in \mathcal{T}_\Lambda$ such that $Q_j(0) = 1$ and such that for every compact set K contained in $(-\infty, 0)$ we have $\sup_{y \in K} |Q_j(y)| \to 0,\ j \to \infty$.*

The proof relies on the following result:

Lemma 4.1 *Let Λ be a closed and discrete set of real numbers. If there exist an interval $I = [a, b]$, a real number $x_0 < a$ and a constant C such that for every $P \in \mathcal{T}_\Lambda$ one has*

$$|P(x_0)| \leq C \sup_{y \in I} |P(y)| \tag{15}$$

then there exist a weight $\omega(x)$ and a compact set K such that (9) *holds for every $x < a$.*

Let $\eta = a - x_0$ and $J = [a, b + \eta]$. Let us fix $u \in [x_0, a]$. The space $\mathcal{T}_\Lambda$ being translation invariant (15) can be applied to $Q(x) = P(x + u - x_0)$. Therefore (15) remains valid when x_0 is replaced by u and I by J. We then proceed inductively from the interval $E_m = [x_0 - m\eta, a - m\eta]$ to E_{m+1}, $m \in \mathbb{N}$. This inductive procedure yields (9) with an exponential weight.

Needless to say Lemma 4.1 is also true if $x_0 < a$ is replaced by $x_1 > b$. The conclusion is the validity of (9) for $x > b$. Finally Lemma 4.1 implies Theorem 4.1. Here is the argument. Let us assume that Λ is wild. Then one of the two following conditions is satisfied: either for any interval $I = [a, b]$ and any $x_0 < a$ there exists a sequence $P_j \in \mathcal{T}_\Lambda$ such that $P_j(x_0) = 1$ and P_j converges to 0 uniformly on I or for any interval $I = [a, b]$ and any $x_1 > b$ there exists a sequence $P_j \in \mathcal{T}_\Lambda$ such that $P_j(x_1) = 1$ and P_j converges to 0 uniformly on I. Everything being translation invariant, we can assume $x_0 = 0$ if the first case occurs. We set $I_m = [m^{-1}, m]$ and there exists a $P_{j_m} \in \mathcal{T}_\Lambda$, such that $P_{j_m}(0) = 1$ and $\sup_{y \in I_m} |P_{j_m}(y)| \leq 2^{-j}$. This sequence P_{j_m}, $m \in \mathbb{N}$, is the sequence announced in Theorem 4.1. The second alternative is similar.

An example is given now. Let $\theta > 2$ be a real number. We define Λ_θ as the set of all finite sum $\sum_{k \geq 0} \epsilon_k \theta^k$, $\epsilon_k \in \{0, 1\}$.

Theorem 4.2 *Let us assume that θ is not a Pisot-Thue-Vijayaraghavan number. Then Λ_θ is wild.*

We consider the sequence $P_m(x)$ of finite products $\prod_0^{m-1}(\frac{1+\exp(2\pi i\theta^k x)}{2})$. The spectrum of P_m is contained in Λ. By Pisot's theorem we know that $|P_m(x)| = \prod_0^{m-1}|\cos(\pi\theta^k x)|$ converge uniformly to 0 on every compact set not containing the origin. We have $P_m(0) = 1$ which concludes the proof.

The converse is true. If θ is a Pisot-Thue-Vijayaraghavan number then Λ_θ satisfies (8) as it is proved in [14].

Here is a second example where the definition of wild sets of frequencies is quite natural. We consider the wave equation on the three dimensional torus $\mathbb{T}^3$.

Theorem 4.3 *For every $T_1 > T_0 > 0$ and every $\epsilon > 0$ there exists a solution $v(x,t)$ of the wave equation on $\mathbb{T}^3$ such that $v(0,0) = 1$ and*

$$|v(x,t)| \le \epsilon,\ T_0 \le t \le T_1,\ x \in \mathbb{T}^3. \tag{16}$$

The proof of this simple observation depends on the following remarks. Let $w(x,t)$ be defined on $\mathbb{T}^3 \times \mathbb{R}$ by

$$w(x,t) = t + \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{\sin(2\pi t|k|)}{2\pi|k|}\exp(2\pi i k\cdot x).$$

Then $w(x,t)$ is the solution to the following Cauchy problem (C-1) for the wave equation on $\mathbb{T}^3 \times \mathbb{R}$

(1) $\frac{\partial^2}{\partial t^2} u(x,t) = \Delta\, u(x,t)$

(2) $u(x,0) = 0,\ \frac{\partial}{\partial t} u(x,0) = \delta_0(x)$.

But $w(x,t)$ can also be computed by periodizing the solution of a similar Cauchy problem (C-2) on $\mathbb{R}^3\times\mathbb{R}$. This scheme is detailed now. Let σ_t, $t\in\mathbb{R}$, be the normalized surface measure on the sphere $B_t \subset \mathbb{R}^3$ centered at 0 with radius $|t|$ (the total mass of σ_t is 1). Then $u(x,t) = t\,\sigma_t(x)$ belongs to $\mathcal{C}^\infty(\mathbb{R},\mathcal{S}'(\mathbb{R}^3))$ and is the solution of the Cauchy problem (C-2):

(i) $\frac{\partial^2}{\partial t^2} u(x,t) = \Delta\, u(x,t)$

(ii) $u(x,0) = 0,\ \frac{\partial}{\partial t} u(x,0) = \delta_0(x)$.

Therefore

$$w(x,t) = \sum_{k\in\mathbb{Z}^3} t\,\sigma_t(x-k) \tag{17}$$

is the solution of the following Cauchy problem (C-1). Let us consider the distribution $\tau(x,t) = \frac{\partial}{\partial t} w(x,t)$. Let$\phi$ be a compactly supported smooth function defined on $\mathbb{R}^3$ and for $\epsilon > 0$ let $\phi_\epsilon(x) = g(x/\epsilon)$. If ϵ is small enough ϕ_ϵ can be viewed as a function defined on $\mathbb{T}^3$. Let $g_\epsilon(x,t)$ be the solution of the wave equation defined by $g_\epsilon = \tau * g_\epsilon$ where the convolution product takes place on $\mathbb{T}^3$. Then $g_\epsilon(0,0) = 1$ while $|g_\epsilon(x,t)| \le C\epsilon$ if $t \in [T_0, T_1]$ as simple estimates show. Therefore the set $\Lambda \subset \mathbb{R}^4$ defined by $\Lambda = \{k, \pm\sqrt{|k|},\ k\in\mathbb{Z}^3\}$ is wild.

5 Unions of lattices

An interesting example is detailed in this section. It will now be assumed that

$$\Lambda = \bigcup_{0}^{\infty} \omega_j \mathbb{Z} \tag{18}$$

where $1 = \omega_0 < \cdots < \omega_j < \cdots$ and $\sum_0^\infty 1/\omega_j < \infty$. Then we have

Lemma 5.1 *If Λ is defined by (18) then $\mathcal{T}_\Lambda$ is not dense in $\mathcal{C}(\mathbb{R}^n)$ for the topology of uniform convergence on compact sets.*

Let $g_j(x)$, $j \in \mathbb{N}$, be defined by $g_j(x) = \pi\omega_j$ on $[-1/(2\pi\omega_j), 1/(2\pi\omega_j)]$ and $g_j(x) = 0$ outside this interval. The convolution products $g_0 * g_1 * \cdots * g_j$ converge to a $\mathcal{C}^\infty$ function g. We have $g \geq 0$, $\int g = 1$ and $\widehat{g} = 0$ on $\Lambda \setminus \{0\}$. The function $h = \frac{d}{dx} g$ has the required properties: h is not identically 0, it is compactly supported, and its Fourier transform vanishes on Λ. This ends the proof.

Theorem 5.1 *Let us furthermore assume that the real numbers $1/\omega_j$, $j \in \mathbb{N}$, are linearly independent over $\mathbb{Q}$. Then the upper density of $\Lambda = \bigcup_0^\infty \omega_j \mathbb{Z}$ is infinite and Λ is wild.*

It will be proved that in full generality the property 'infinite upper density' implies 'wild'. Here the two properties will be proved by the same argument. We argue by contradiction and assume that (9) holds. The proof of Theorem 5.1 begins with the following definition.

Definition 5.1 *Let $\Lambda \subset \mathbb{R}$ be a closed and discrete set and let F be a finite set. We write $F \in \mathcal{F}(\Lambda)$ if there exists a sequence x_j, $j \in \mathbb{N}$, of real numbers such that $F + x_j \subset \Lambda + [-1/j, 1/j]$.*

If $|x_j| \leq C$, $F \in \mathcal{F}(\Lambda)$ simply means $F \subset \Lambda + a$ for some a. If $\Lambda = \{k + 2^{-k}, k \in \mathbb{N}\}$ then $F \subset \mathbb{Z} \Rightarrow F \in \mathcal{F}(\Lambda)$. The proof of Theorem 5.1 depends on the following lemma:

Lemma 5.2 *Let us assume that (9) holds for a pair (Λ, K) and for a weight ω. If $F \in \mathcal{F}(\Lambda)$ (9) also holds for the pair (F, K) and the same weight ω.*

Let $P(x) = \sum_{y \in F} c(y) \exp(2\pi i \lambda y)$ be an arbitrary trigonometric sum with frequencies in F. We want to prove that

$$|P(x)| \leq \omega(x) \sup_{y \in K} |P(y)|. \tag{19}$$

For $j \geq 1$ every $y \in F$ can be written as $y = \lambda_{j,y} - x_j + \epsilon_j$ where $|\epsilon_j| \leq 1/j$. We approach $P(x)$ by $P_j(x) = \exp(-2\pi i x_j x) Q_j(x)$ where $Q_j(x) = \sum_{y \in F} c(y) \exp(2\pi i \lambda_{y,j} x)$. But (9) is true for Q_j by assumption. Since $|P_j| = |Q_j|$ (9) is also true for P_j and it suffices to let j tend to infinity to conclude the proof of Lemma 4.2.

Lemma 5.3 *The finite set $F_{\epsilon,n} = \{0, \epsilon, 2\epsilon, \cdots, (n-1)\epsilon\}$ belongs to $\mathcal{F}(\Lambda)$ for every $\epsilon > 0$ and every integer n.*

Lemma 5.3 obviously implies that the upper density of Λ is infinite. This upper density is defined as

$$\limsup_{T\to\infty} T^{-1} \sup_{x\in\mathbb{R}} \#([x, x+T] \cap \Lambda). \tag{20}$$

We return to the proof of Lemma 5.3. Here the linear independence of $1/\omega_m$, $m \in \mathbb{N}$, over $\mathbb{Q}$ is used. The subgroup

$$(\exp(2\pi ik/\omega_m)_{0\le m\le n-1},\ k \in \mathbb{Z},$$

is dense in $\mathbb{T}^n$. Therefore there exists a sequence k_j of integers and n sequences $l_{j,m}$, $j \in \mathbb{Z}$, $1 \le m \le n$, of integers such that for $1 \le m \le n$, $k_j/\omega_m - l_{j,m} + m\epsilon/\omega_m \to 0$, $j \to \infty$. This convergence takes place on the real line. It obviously implies $k_j - \omega_m l_{j,m} + m\epsilon \to 0$, $j \to \infty$. We set $x_j = k_j$ and we have $m\epsilon = \lim_{j\to\infty} \omega_m l_{j,m} - x_j$ as announced.

We now disprove the uniform validity of (9) when Λ is replaced by $F_{\epsilon,m}$ and $\epsilon \to 0$, $m \to \infty$. To this end we form $P_{\epsilon,m} = \epsilon^{-m}\sum_0^{m-1} c_k \exp(2\pi ikx)$ where $\sum_0^{m-1} c_k k^q = 0$, $0 \le q \le m-1$. Lemma 4.2 implies that (9) is satisfied by $P_{\epsilon,m}$ uniformly with respect to ϵ and m. But $\lim_{\epsilon\to 0} P_{\epsilon,m} = cx^m$. Therefore (9) is satisfied by x^m uniformly in $m \in \mathbb{N}$ which is impossible. The same argument can be used to disprove (10).

If Λ is replaced by a finite union $\Lambda_N = \bigcup_0^N \omega_j\mathbb{Z}$ then (9) is satisfied with $\omega(x) = (1+|x|)^N$ as it is proved in [17].

6 Upper densities

Theorem 5.1 is a special instance of a more general fact.

Theorem 6.1 *Let Λ be a closed and discrete set of real numbers. Then property (9) implies that the upper density of Λ is finite.*

We argue by contradiction. Property (9) implies the following: There exist an interval $[a, b]$ where $0 < a < b$ and a Radon measure ν carried by $[a, b]$ such that $\mu = \delta_0 - \mu$ satisfies

$$\widehat{\mu} = 0 \text{ on } \Lambda. \tag{21}$$

Lemma 6.1 *Property (21) implies that the upper density of Λ is finite.*

Lemma 6.1 relies the following corollary of Jensen's formula.

Lemma 6.2 *Let F be an entire function in the complex plane such that for a constant $T > 0$ and every $z = x + iy$ we have $|F(z)| \le \exp(T|y|)$. Let $I = [u, v]$ be an interval of length $r = v - u$ and assume that F has N zeros on I. Let $J = [u-r, u]$ and $J' = [v, v+r]$. Then*

$$\sup_{x\in J} |F(x)| \le C \exp[Tr/\pi - N\log(3/2)]. \tag{22}$$

To establish this result let us freeze $x_0 \in J$ and prove (22) when $x = x_0$. If $F(x_0) = 0$ there is nothing to prove. We can assume $F(x_0) \neq 0$. Then Lemma 6.2 follows from Jensen's formula applied to the entire function $G(z) = F(z+x_0)$. Let us assume that G does not vanish on the circle $|z| = R = 3r$. Then Jensens's formula yields

$$\frac{1}{2\pi}\int_0^{2\pi} \log|G(R\exp(i\theta))|\,d\theta - \log|G(0)| = \int_0^R n(t)\,dt/t \tag{23}$$

where $n(t)$ is the number of zeros of G in the disk $|z| \leq t$. This identity is now applied to $R = 3r$. We assumed that G does not vanish on the circle $|z| = R$. If G does vanish R shall be replaced by a suitable $R' \in [3r, 4r]$ in what follows. It will not change the required estimates. A minorant of $\int_0^R n(t)\,dt/t$ is given by $\log(3/2)N$ since F has N zeros on I. A majorant of $\frac{1}{2\pi}\int_0^{2\pi} \log|G(R\exp(i\theta)|\,d\theta$ is rT/π. Then (22) follows from (23).

We now return to the proof of Theorem 6.1. Let

$$F(z) = \int \exp(-2\pi izt)\,d\mu(t) \tag{24}$$

the Fourier-Laplace transform of μ. We have

$$|F(z)| \leq \|\mu\| \exp[2\pi b|y|]. \tag{25}$$

If Λ is a closed and discrete set with an infinite upper density there exists a sequence I_m of intervals enjoying the following properties

(a) $|I_m| \to \infty$ with m.

(b) the number of zeros of F on I_m exceeds $m|I_m|$.

Then (22) implies

$$\sup|F(x)| \leq \exp(6b|I_m| - \log(3/2)m|I_m|) \leq \exp(-cm|I_m|) \tag{26}$$

with $c > 0$. But $F(x) = 1 - \widehat{\nu}(x)$ and (26) implies

$$\frac{1}{|I_m|}\int_{J_m} |1 - \widehat{\nu}(x)|\,dx \to 0,\; m \to \infty. \tag{27}$$

Let us decompose ν into an atomic measure σ and a continuous measure ρ. Then (27) implies $\widehat{\rho} = 1$ (since $\widehat{\rho}$ is an almost periodic function) and $\rho = \delta_0$. Here we reach the required contradiction since ρ is carried by $[a, b]$ and $0 \notin [a, b]$.

7 Minimal sets

Let us return to the problem of minimal sets K satisfying Property 1.2 when Λ is given. If $\Lambda = \mathbb{Z}$ then $K = [0, 1]$ is minimal. If Λ is a simple quasi-crystal with density d then (8) holds for every Riemann integrable compact set K whose Lebesgue measure satisfies $|K| > d$. It will be shown that this property fails if $|K| = d$. For proving this observation let us begin by defining a simple quasi-crystal. Let $m, n \in \mathbb{N}$, $N = m + n$, and $\Gamma \subset \mathbb{R}^N$ be a lattice: $\Gamma = A(\mathbb{Z}^N)$ where $A \in \mathbb{G}L_N$. For $(x, t) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$, we write $p_1(x, t) = x$, $p_2(x, t) = t$. Let us assume that p_1 once restricted to Γ is a 1-1 mapping with a dense range. The same is required from p_2.

Definition 7.1 *Let $I \subset \mathbb{R}^m$ be a Riemann integrable compact set (a window) with a positive measure. Then the model set $\Lambda_I \subset \mathbb{R}^n$ is defined by*

$$\Lambda_I = \{p_1(\gamma); \gamma \in \Gamma,\, p_2(\gamma) \in I\}. \tag{28}$$

This model set is a simple quasi-crystal if $m = 1$ and if I is an interval.

Theorem 7.1 *Let $\Lambda_I \subset \mathbb{R}^n$ be a simple quasi-crystal and $K \subset \mathbb{R}^n$ be a Riemann integrable compact set. Then the following two conditions are equivalent:*

- (a) *Property 1.2 is satisfied by the pair Λ_I, K.*
- (b) $|K| > \operatorname{dens} \Lambda_I$.

This result sharply contrasts with Theorem 2.1. We do not know what happens in the L^p case. The proof of $(b) \Rightarrow (a)$ can be found in [16]. The proof of $(a) \Rightarrow (b)$ relies on the duality principle which was used in [11]. Let $P(x) = \sum_{\gamma \in \Gamma} c(\gamma) \exp(2\pi i x \cdot p_1(\gamma))$ be an arbitrary trigonometric sum whose frequencies belong to Λ_I. Since $K \cap p_1(\Gamma^*)$ is dense in K we have

$$\sup_{x \in K} |P(x)| = \sup_{x \in K \cap p_1(\Gamma^*)} |P(x)|. \tag{29}$$

Here comes the duality principle: we have $p_1(\gamma) \cdot p_1(\gamma^*) + p_2(\gamma) p_2(\gamma^*) \in \mathbb{Z}$ for every $\gamma \in \Gamma$ and every $\gamma^* \in \Gamma^*$. It implies $P(x) = Q(\overline{x})$ for every $x \in K \cap p_1(\Gamma^*)$. Here $x = p_1(\gamma^*), \overline{x} = p_2(\gamma^*)$ and $Q(y) = \sum_{\gamma \in \Gamma; p_2(\gamma) \in I} c(\gamma) \exp(2\pi i y p_2(\gamma))$. Let M_I be the set of all $t = p_2(\gamma^*)$ with $p_1(\gamma^*) \in K$. We obviously have $\|P\|_\infty = \|Q\|_\infty$ and (8) is equivalent to

$$\|Q\|_\infty \le C \sup_{t \in M_I} |Q(t)|. \tag{30}$$

Observe that the frequencies of the trigonometric sum Q belong to I. Since $p_2(\Gamma)$ contains arbitrarily fine grids (30) is equivalent to the same condition where Q is replaced by an arbitrary function in the Paley-Wiener space PW_I. The space PW_I consists of the functions $f \in L^2(\mathbb{R})$ whose Fourier transform is supported by I. Therefore the validity of (30) depends on a famous theorem by Beurling [3] on "balayage". A necessary and sufficient condition for the validity of (30) is $\operatorname{dens} M_K > |I|$ which implies $|K| > \operatorname{dens} \Lambda_I$ as announced.

8 L^p-estimates

In a special situation we can answer the problem raised by Property 4 of Section 1. Once for all $\Lambda \subset \mathbb{R}^n$ is a uniformly discrete set and $\sigma_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$.

Definition 8.1 *We say that Λ is a gentle set if the distributional Fourier transform $\widehat{\sigma_\Lambda}$ of σ_Λ is a Radon measure.*

A lattice is a gentle set. A finite union of gentle sets is a gentle set. But the set Λ defined by (18) is not a gentle set. The calculation of the Fourier transform of σ_Λ is an amusing exercise. The set $\Lambda_{\alpha,\beta}$ which was defined in Section 2 is a gentle set as it was proved in [17]. In this case $\widehat{\sigma_\Lambda}$ is an atomic measure. This fact will be used later on.

A gentle set has a finite upper density which explains why the set defined by (18) is not a gentle set. Indeed let ϕ be a compactly supported continuous function whose Fourier transform $\widehat{\phi}$ is non negative. Then for every $y \in \mathbb{R}^n$ we have $\int \widehat{\phi}(y-x)\, d\sigma_\Lambda(x) = \int \exp(2\pi i y\cdot u)\phi(u)\, d\widehat{\sigma}_\Lambda(u) = I(y)$ and $|I(y)| \le C$ since $\widehat{\sigma}_\Lambda$ is a Radon measure.

Let Λ be a gentle set. Then $\mu = \widehat{\sigma_\Lambda}$ is a Radon measure. We set $w(y) = \int_{B_y} d|\mu|$, $y \in \mathbb{R}^n$, where B is the ball centered at 0 with radius 1 and $B_y = B+y$. The following theorem was proved in [17]:

Theorem 8.1 *Let Λ be a gentle set. Let us assume that w has a polynomial growth at infinity and let $\omega \ge 1$ be a continuous and sub-multiplicative function which is a majorant of w. Then there exists a compact set K such that for every $f \in \mathcal{C}_\Lambda$ we have*

$$\forall y \in \mathbb{R}^n,\ |f(y)| \le \omega(y) \sup_{u\in K} |f(u)|. \tag{31}$$

This estimate is optimal if the difference between w and ω is forgotten. Indeed we let g be a continuous function supported by the unit ball and normalized by $|g\|_\infty = 1$. We consider the convolution product $f = \mu * g$. This function belongs to $\mathcal{C}_\Lambda$ and (31) is satisfied. For a given x we have

$$|\int g(y-x)\, d\mu(y)| \le C'\omega(x) \tag{32}$$

where C' is the total mass of μ on K. We now take the supremum of the LHS with respect to g and obtain $|\mu|(B_x) \le C'\omega(x)$.

The L^p version of this theorem is given now.

Theorem 8.2 *Let Λ be a gentle set. Let us assume that w has a polynomial growth at infinity and let $\omega \ge 1$ be a continuous and sub-multiplicative function which is a majorant of w. Let $1 \le p \le \infty$. Then there exists a compact set K such that for every $f \in \mathcal{C}_\Lambda$ and every $y \in \mathbb{R}^n$ we have*

$$(\int_{K+y} |f(x)|^p\, dx)^{1/p} \le C\omega(y)^{|2-p|/p}(\int_K |f(x)|^p\, dx)^{1/p}. \tag{33}$$

If $p = \infty$ this is Theorem 8.1. If $p = 2$ this is a trivial statement. Theorem 8.2 will result by interpolation between these two cases as indicated by Lemma 8.2. To prepare the ground for the proof we define $\beta > 0$ by

$$\inf_{\{\lambda \neq \lambda',\, \lambda, \lambda' \in \Lambda\}} |\lambda - \lambda'| = \beta > 0.$$

Let $0 < r < r' < \beta/2$, let B_r (resp. B'_r) the ball centered at 0 with radius r (resp. r'). Let ϕ be a function in the Schwartz class $\mathcal{S}$ such that $\widehat{\phi} = 1$ on B_r and $\widehat{\phi} = 0$ outside B'_r. Let μ_y be the Radon measure μ translated by $-y$ and let $\chi_y(x) = \exp(2\pi ixy)$. Then the Fourier transform of the product $\phi\mu_y$ is the convolution product $\widehat{\phi} * \chi_y \sigma_\Lambda$. The following lemma resumes this discussion:

Lemma 8.1 *We have*

$$\widehat{\phi\mu_y}(\xi) = \sum_{\lambda \in \Lambda} \exp(2\pi i\lambda \cdot y)\, \widehat{\phi}(\xi - \lambda). \tag{34}$$

We now estimate the norm of the measure $\phi\mu_y$. We have

$$\|\phi\mu_y\| \leq C\omega(y). \tag{35}$$

This estimate results from the definition of w, the rapid decay of ϕ, and the slow growth of w.

The operator norm of the convolution with the measure $\nu_y = \phi\mu_y$ acting on $L^\infty(\mathbb{R}^n)$ does not exceed $\|\nu_y\| \leq C\omega(y)$. The same bound is valid on $L^1(\mathbb{R}^n)$. On the other hand this convolution operator acts on $L^2(\mathbb{R}^n)$ with a norm not exceeding C. Indeed (34) shows that $\|\widehat{\nu}_y\|_\infty \leq C$ uniformly in y. An interpolation between L^2 and L^∞ or L^1 yields the following:

Lemma 8.2 *Let $p \in [1, \infty]$. Then we have, for every $y \in \mathbb{R}^n$ and every $f \in L^p$,*

$$\|\nu_y * f\|_p \leq C\omega(y)^{|2/p-1|} \|f\|_p\,. \tag{36}$$

Let g be a positive function in the Schwartz class whose Fourier transform is supported by the ball centered at 0 with radius r. Let $P \in \mathcal{T}_\Lambda$ and let us set $P_y(x) = P(x + y)$.

Lemma 8.3 *For every $y \in \mathbb{R}^n$ we have*

$$P_y g = \nu_y * (Pg). \tag{37}$$

To prove this lemma it suffices to do it when $P(x) = \exp(2\pi i\lambda \cdot x)$, $\lambda \in \Lambda$. Then the Fourier transform of the LHS of (37) is $\exp(2\pi i\lambda \cdot y)\widehat{g}(\xi - \lambda)$ while the Fourier transform of the RHS is $\exp(2\pi i\lambda \cdot y)\widehat{\phi}(\xi - \lambda)\widehat{g}(\xi - \lambda)$. But $\widehat{\phi} = 1$ on the support of $\widehat{g}$ which ends the proof of Lemma 8.3.

We now return to Theorem 8.2 and to the proof of (33). For simplifying the notations let us set $\omega_p(y) = \omega(y)^{|2/p-1|}$. Then

Lemma 8.4 *For every $P \in \mathcal{T}_\Lambda$, every $y \in \mathbb{R}^n$, and every $R \geq 1$ we have*

$$(\int_{|x-y|\leq R} |P(x)|^p \, dx)^{1/p} \leq C_R \, \omega_p(y) \|Pg\|_p \, . \tag{38}$$

We have by (37)

$$\|P_y g\|_p = \|\nu_y * (Pg)\|_p \leq C\omega_p(y)\|Pg\|_p \, . \tag{39}$$

We have (39) $\Rightarrow$ (38). Indeed it suffices to observe that $g(x) \geq c_R > 0$ on the ball centered at 0 with radius R.

If g was compactly supported (38) would end the proof of Theorem 8.2. This is not the case but the problem can be easily fixed since g has a rapid decay at infinity. We now give the details of this argument.

Lemma 8.5 *Let Q_T be the cube defined by $|x_1| \leq T, \ldots, |x_n| \leq T$, and let $\mathcal{R}_T = \mathbb{R}^n \setminus Q_T$. For every $\epsilon > 0$ there exists an integer $T \geq 1$ such that for every $P \in \mathcal{T}_\Lambda$ we have*

$$(\int_{\mathcal{R}_T} |Pg|^p \, dx)^{1/p} \leq \epsilon \|Pg\|_p \, . \tag{40}$$

For proving this estimate we pave $\mathcal{R}_T$ by a disjoint union of cubes Q^j, $j \in \mathbb{N}$, of size 1. Then (38) implies $\int_{Q^j} |P|^p \, dx \leq C\omega_p^p(x_j)\|Pg\|_p^p$ where x_j is the center of Q^j. Therefore

$$\int_{Q^j} |Pg|^p \, dx \leq C\omega_p^p(x_j) \sup_{x\in Q_j} |g(x)|^p \|Pg\|_p^p \, . \tag{41}$$

We add these estimates which yields $\int_{\mathcal{R}_T} |Pg|^p \, dx \leq C \sum_j \omega_p^p(x_j) \sup_{Q_j} |g|^p \|Pg\|_p^p \leq \epsilon^p \|Pg\|_p^p$.

Corollary 8.1 *The three norms $(\int_{Q_T} |Pg|^p \, dx)^{1/p}$, $(\int_{Q_T} |P|^p \, dx)^{1/p}$, and $\|Pg\|_p$ are equivalent on $\mathcal{T}_\Lambda$ if T is large enough.*

Finally this corollary and (38) imply Theorem 8.2.

In a special case (Theorem 8.4) these estimates are optimal. Theorem 8.4 follows from a general result which is given now. Let us assume that $\widehat{\sigma_\Lambda}$ is the atomic measure $\mu = \sum_0^\infty a_j \delta_{x_j}$ and let $\omega_{p,R}(x) = (\sum_{|x-x_j|\leq R} |a_j|^p)^{1/p}$. We further assume that there exists a constant C such that $\omega_{p,2R}(x) \leq C\omega_{p,R}(x)$ holds for $R \geq 1$. We write $\omega_{p,1} = \omega_p$.

Theorem 8.3 *For every $y \in \mathbb{R}^n$ and for every $R \geq 1$ there exists a non trivial $f \in \mathcal{C}_\Lambda$ such that*

$$(\int_{|x-y|\leq 1} |f(x)|^p \, dx)^{1/p} \geq C_R \, \omega_p(y) (\int_{|x|\leq R} |f(x)|^p \, dx)^{1/p} \, . \tag{42}$$

Let us observe that $\omega_2 \simeq 1$ since Λ is uniformly discrete. It implies $\omega_p \leq C$ if $p \geq 2$. Moreover if $1 \leq p \leq 2$ Hölder's inequality implies

$$\omega_{p,R}(x) = (\sum_{|x-x_j|\leq R} |a_j|^p)^{1/p} \leq (\sum_{|x-x_j|\leq R} |a_j|)^{(2-p)/p}(\sum_{|x-x_j|\leq R} |a_j|^2)^{(p-1)/p}.$$

It shows that $\omega_{p,R}(x) \leq C\omega(x)^{(2-p)/p}$. Therefore (33) and (42) are compatible. But there is a gap between the upper bound given by (33) and the lowed bound given by (42). In some exceptional cases $\omega_{p,R}(x) \simeq \omega(x)^{(2-p)/p}$. An example is given below (Theorem 8.4). The proof mimics the argument used in the proof of Lemma 6.5 in [11]. Let $\epsilon > 0$ and let ϕ an even compactly supported smooth function. We define $\phi_{\epsilon,p}(x) = \epsilon^{-n/p}\phi(x/\epsilon)$ and we have $\|\phi_{\epsilon,p}\|_p = \|\phi\|_p$. We consider the convolution product $f_{\epsilon,p} = \mu * \phi_{\epsilon,p}$. This function belongs to $\mathcal{C}_\Lambda$ and is our candidate to obtain (42). Local L^p norms of $f_{\epsilon,p}$ are computed as follows:

Lemma 8.6 *Let a_j, $j \in \mathbb{N}$, be a sequence in l^1 and let $x_j \in \mathbb{R}^n$ be a sequence of pairwise disjoint points. Let K be a Riemann integrable compact set whose boundary ∂K does not contain any x_j. Then for $1 \leq p \leq \infty$ we have*

$$\lim_{\epsilon\to 0}(\int_K |\sum_j a_j\phi_{\epsilon,p}(x-x_j)|^p\,dx)^{1/p} = (\sum_{x_j\in K} |a_j|^p)^{1/p}. \tag{43}$$

Given $\eta > 0$ one fixes N such that $\sum_{N+1}^\infty |a_j| \leq \eta$. The triangle inequality implies $(\int_K |\sum_{N+1}^\infty a_j\phi_{\epsilon,p}(x-x_j)|^p\,dx)^{1/p} \leq \eta$. Next $\epsilon_N > 0$ is fixed such that the supports of $\phi_{\epsilon,p}(x-x_j)$, $x_j \in K, 0 \leq j \leq N, 0 < \epsilon \leq \epsilon_N$, are pairwise disjoint. Then

$$(\int_K |\sum_0^N a_j\phi_{\epsilon,p}(x-x_j)|^p\,dx)^{1/p} = (\sum_{x_j\in K,\, 0\leq j\leq N} |a_j|^p)^{1/p}. \tag{44}$$

This ends the proof of Lemma 8.6.

If one the condition $x_j \notin \partial K$ is dropped (43) shall be replaced by

$$\lim_{\epsilon\to 0}(\int_K |\sum_j a_j\phi_{\epsilon,p}(x-x_j)|^p\,dx)^{1/p} \geq (\sum_{x_j\in L} |a_j|^p)^{1/p}. \tag{45}$$

where L is any compact set contained in the interior of K. Finally (45) implies Theorem 8.3.

We illustrate these theorems by the one dimensional example of the set $\Lambda_{\alpha,\beta}$ which was defined in Section 2. In this case the Fourier transform of the measure $\sigma_{\Lambda_{\alpha,\beta}}$ is an explicit atomic measure [17]. Then Theorem 8.2, Theorem 8.3, and the explicit calculation in [17] imply the following result:

Theorem 8.4 *Let $1 \leq p \leq \infty$ and $\omega_p(x) = C(1+|x|)^{|1/p-1/2|}$. Then for every $f \in \mathcal{C}_{\Lambda_{\alpha,\beta}}$ we have for every y*

$$(\int_{y-1}^{y+1} |f(x)|^p \, dx)^{1/p} \leq \omega_p(y)(\int_{-1}^{1} |f(x)|^p \, dx)^{1/p} \tag{46}$$

and this estimate is optimal if $1 \leq p \leq 2$.

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